Optimistic multi-granulation fuzzy rough set model in ordered information system

Weihua Xu
School of Mathematics and Statistics
Chongqing University of Technology
Chongqing, P.R.China 400054
Email: chxuwh@gmail.com

Wenxin Sun
School of Mathematics and Statistics
Chongqing University of Technology
Chongqing, P.R.China 400054
Email: sunxuxin520@163.com

Yufeng Liu
School of Mathematics and Statistics
Chongqing University of Technology
Chongqing, P.R.China 400054
Email: liyufeng@cqut.edu.cn

***Abstract*—With granular computing point of view, the classical dominance based rough set model is based on a single granulation. On the basis of the analysis of rough set model over a dominance relation and the theory of fuzzy rough set, a new generalized rough set model over dominance relations is constructed, which is optimistic multi-granulation fuzzy rough set model over dominance relations. It follows the research on the properties of the lower and upper approximations of the optimistic multi-granulation fuzzy rough set model over dominance relations. The fuzzy rough set model and classical rough set model on a dominance relation are special cases of the new one from the perspective of the considered concept and granular computing.**

***Keywords*-multi-granulation; rough set; fuzzy rough set; dominance relations**

I. Introduction

Rough set theory, proposed by Pawlak [7], has been applied successfully in the fields of pattern recognition, medical diagnosis, data mining, conflict analysis, algebra. In recent years, the generalization of classical rough set model is one of the most important study spotlights.

It is widely acknowledge that the classification of objects in the classical approximation space is based on the approximation classification of equivalence relation. Moreover, rough set theory was also discussed with the point view of granular computing. Information granules refers to pieces, classes and groups divided in accordance with characteristics and performances of complex information in the process of human understanding, reasoning and decision-making. Zadeh firstly proposed the concept of granular computing and discussed issues of fuzzy information granulation in 1979 [5]. In 1985, Hobbs proposed the concept of granularity [3]. Then the basic idea of information granulation had been applied to many fields including rough set. In the point view of granulation computing which played a more and more important role gradually in soft computing [6], knowledge discovery, data mining and many excellent results were achieved [9], the classical Pawlak rough set is based on a single granulation induced from an equivalence relation. However, when the rough set is based on many granulations induced from several indiscernibility relations, we can have some cases as follow:

Case 1. There exists a granulation at least such that the elements surely belong to the concept.

Case 2. There are some granulations such that the elements surely belong to the concept.

Case 3. All of the granulations such that the elements surely belong to the concept.

Case 4. There exists a granulation at least such that the elements possibly belong to the concept.

Case 5. There are some granulations such that the elements possibly belong to the concept.

Case 6. All of the granulations such that the elements possibly belong to the concept.

For the above of these cases, Qian extended the Pawlak rough set to multi-granulation rough set models where the approximation operators are defined by multiple equivalence relations on the universe [8]. On the basic, many researchers have extended the multi-granulation rough set to the generalize multi-granulation rough sets [4], [11]–[13].

Besides, rough set theory can be generalized by combining with other theories that deal with uncertainty knowledge such as fuzzy set. It has been acknowledged by different studies that fuzzy set theory and rough set theory are complementary with handling different kinds of uncertainty. Dubois and Prade proposed concepts of rough fuzzy sets and fuzzy rough sets based on approximations of fuzzy sets in crisp approximations spaces, and crisp sets in fuzzy approximation spaces, respectively [1]. From then on, more and more researches began to extend the rough set by combining rough sets and the theory of fuzzy sets.

Moreover, in many real situations, we often face the problems about the the ordering of objects. For this reason, many researchers [2], [10] proposed an extended rough set theory, called the dominance-based rough set approach(DRSA) to take into account the ordering properties of criteria. In this paper, we will propose a kind of multi-granulation fuzzy rough set model over dominate relations by combining rough set and fuzzy set with the point view of granulation computing. The rest of this paper is organized as follows. Some preliminary concepts of dominance rough set theory and fuzzy rough set theory are showed in Section 2. In Section 3, for a fuzzy target information system, based on multiple dominate relations, the optimistic multi-granulation

978-1-4673-2311-6/12/$31.00 ©2012 IEEE

fuzzy rough approximation operators of a fuzzy concept are constructed and a number of important properties are discussed in detail. Finally, the paper is concluded by a summary for in Section 4.

II. Preliminaries

In this section, we will first review some basic concepts and notions in the theory of rough set over dominance relation and fuzzy rough set on the basis of equivalence relations. More can be found in reference [14].

A. Rough set theory and ordered information system

The following recalls necessary concepts and preliminaries required in the sequel of our work. Detailed description has also been made in [14], [15]. The notion of information system (sometimes called data tables, attribute valued systems, knowledge representation systems, etc.) provides a convenient basis for the representation of objects in terms of their attributes.

An information system is a quadruple $\mathcal{I} = (U, AT, V, f)$, where U is a non-empty finite set with n objects, $\{u_1, u_2, \ldots, u_n\}$, called the universe of discourse; $AT = \{a_1, a_2, \ldots, a_m\}$ is a non-empty finite set with m attributes; $V = \bigcup\limits_{a \in AT} V_a$ and V_a is the domain of attribute a; $f : U \times AT \longrightarrow V$ is a function such that $f(u, a) \in V_a$ for any $a \in AT$, $u \in U$, called an information function. A decision table is a special case of an information system in which we called the decision as decision attribute, and the others are called condition attributes, to distinguish these attributes. Therefore, $\mathcal{I} = (U, C \bigcup \{d\}, V, f)$ be a decision table, where set C and $\{d\}$ be condition attributes set and decision attribute set respectively.

Assumed that the domain of a criterion $a \in AT$ is complete pre-ordered by an outranking relation $\succcurlyeq_a$, then $u \succcurlyeq_a v$ means that u is at least as good as (outranks) v with respect to the criterion a, and we can say that u dominates v or v is dominated by u. Being of type gain, that is $u \succcurlyeq_a v \Longleftrightarrow f(u, a) \geq f(v, a)$(according to increasing preference) or $u \succcurlyeq_a v \Longleftrightarrow f(u, a) \leq f(v, a)$ (according to decreasing preference). Without any loss of generality and for simplicity, in the following we only consider condition attributes with increasing preference.

For a subset of attributes $A \subseteq AT$, we define $u \succcurlyeq_A v \Longleftrightarrow u \succcurlyeq_a v$ for $\forall a \in A$. That is, u dominates v with respect to all attributes in A. In general, we denote an ordered information system by $\mathcal{I}^{\succcurlyeq} = (U, AT, V, f)$.

For a given ordered information system, we say that u dominates v with respect to $A \subseteq AT$ if $u \succcurlyeq_A v$, and denote by $uR_A^{\succcurlyeq}v$. That is

$$\begin{aligned} R_A^{\succcurlyeq} =& \{(u, v) \in U \times U \mid u \succcurlyeq_A v\} \\ =& \{(u, v) \in U \times U \mid f(u, a) \geq f(v, a) \ \forall a \in A\}, \end{aligned}$$

$R_A^{\succcurlyeq}$ are called a dominance relation of ordered information system $\mathcal{I}^{\succcurlyeq}$.

Let denote

$$\begin{aligned} [u_i]_A^{\succcurlyeq} =& \{u_j \in U \mid (u_j, u_i) \in R_A^{\succcurlyeq}\} \\ =& \{u_j \in U \mid f(u_j, a) \geq f(u_i, a) \ \forall a \in A\}, \\ U/R_A^{\succcurlyeq} =& \{[u_1]_A^{\succcurlyeq}, [u_2]_A^{\succcurlyeq}, \ldots, [u_n]_A^{\succcurlyeq}\}, \end{aligned}$$

where $i \in \{1, 2, \ldots, n\}$, then $[u_i]_A^{\succcurlyeq}$ will be called a dominance class or the granularity of information, and $U/R_A^{\succcurlyeq}$ be called a classification of U about attribute set A.

For any subset $X \subseteq U$ and $A \subseteq AT$ in $\mathcal{I}^{\succcurlyeq}$, the lower and upper approximation of X with respect to a dominance relation $R_A^{\succcurlyeq}$ could be defined as

$$\begin{aligned} \underline{R_A^{\succcurlyeq}}(X) &= \{u \in U \mid [u]_A^{\succcurlyeq} \subseteq X\}, \\ \overline{R_A^{\succcurlyeq}}(X) &= \{u \in U \mid [u]_A^{\succcurlyeq} \cap X \neq \emptyset\}. \end{aligned}$$

B. Fuzzy set and fuzzy rough set

We will first introduce some basic concepts of fuzzy set. Let U be a finite and non-empty set called universe. A fuzzy set A is a mapping from U into the unit interval $[0,\ 1] : \mu : \to [0,\ 1]$, where each $x \in U$ is the membership degree of x in A. Practically, we may consider U as a set of objects of concern and crisp subset of U represents a "non-vague" concept imposed on objects in U. Then a fuzzy set A of U is thought of as a mathematical representation of "vague" concept described linguistically. The set of all the fuzzy sets defined on U is denoted by $F(U)$.

Let A and B be two fuzzy sets on U, the operation between them are defined as

$$\begin{aligned} &(A \cup B)(x) = \max\{A(x), B(x)\}, \\ &(A \cap B)(x) = \min\{A(x), B(x)\}, \\ &\sim A(x) = 1 - A(x), \\ &A \subseteq B \Leftrightarrow A(x) \leq B(x) \ (x \in U). \end{aligned}$$

Let A be a fuzzy set on U, for any $\alpha \in [0, 1]$, if denote

$$\begin{aligned} A_\alpha &= \{x \in U \mid A(x) \geq \alpha\}, \\ A_{\alpha+} &= \{x \in U \mid A(x) > \alpha\}. \end{aligned}$$

then A_α is the α-cut set of A, and $A_{\alpha+}$ is called the strong α-cut set of A.

Let U be the universe, R be an equivalence relation, for a fuzzy set A on U, if take

$$\begin{aligned} \underline{R}(A)(x) &= \wedge\{A(y) | y \in [x]_R\}, \\ \overline{R}(A)(x) &= \vee\{A(y) | y \in [x]_R\} \end{aligned}$$

then $\underline{R}(A)$ and $\overline{R}(A)$ are called the lower and upper approximation of the fuzzy set A with respect to the relation R, where "$\wedge$" means "min" and "$\vee$" means "max" and $[x]_R$ is the equivalence class of x with respect to equivalence relation R. A is a fuzzy definable set if and only if A satisfies $\underline{R}(A) = \overline{R}(A)$. Otherwise, A is called a fuzzy rough set.

Because of the limitation of the paper length, the properties of the above set approximation can be found in reference [14].

III. Optimistic multi-granulation fuzzy rough set in ordered information systems

In this section, we will make researches about optimistic multi-granulation fuzzy rough set which are on the problem of the rough approximations of a fuzzy set over multiple dominate relations.

At first, we will propose a fuzzy rough set model also called single-granulation fuzzy rough set model over a dominate relation in the following [14].

Let $\mathcal{I}^{\succcurlyeq} = (U, AT, V, f)$ be an ordered information system and $A \subseteq AT$. For the fuzzy set $X \in F(U)$, denote

$$\underline{R_A^{\succcurlyeq}}(X)(u) = \wedge\{X(v) \mid v \in [u]_A^{\succcurlyeq}\},$$
$$\overline{R_A^{\succcurlyeq}}(X)(u) = \vee\{X(v) \mid v \in [u]_A^{\succcurlyeq}\},$$

where " $\vee$ " means "max" and " $\wedge$ " means "min", then $\underline{R_A^{\succcurlyeq}}(X)$ and $\overline{R_A^{\succcurlyeq}}(X)$ are the lower and upper approximation of the fuzzy set X over the dominate relation $R_A^{\succcurlyeq}$ with respect to the attribute set A. If $\underline{R_A^{\succcurlyeq}}(X) \neq \overline{R_A^{\succcurlyeq}}(X)$, then the fuzzy set X is a fuzzy rough set over the dominate relation. We can easily find that this model will be the fuzzy rough set model if the above relation R is an equivalence relation.

In the following, we will introduce the optimistic multi-granulation fuzzy rough set (in brief OMGFRS) over dominate relations and its corresponding properties.

Definition 3.1 Let $\mathcal{I}^{\succcurlyeq} = (U, AT, V, f)$ be an ordered information system, $A_1, A_2, \cdots, A_m \subseteq AT$. For the fuzzy set $X \in F(U)$, denote

$$\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}(X)(u) = \bigvee_{i=1}^{m}\{\bigwedge\{X(v) \mid v \in [u]_{A_i}^{\succcurlyeq}\}\},$$

$$\overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}(X)(u) = \bigwedge_{i=1}^{m}\{\bigvee\{X(v) \mid v \in [u]_{A_i}^{\succcurlyeq}\}\},$$

where "$\bigvee$" means "max" and "$\bigwedge$" means "min", then $\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}(X)$ and $\overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}(X)$ are respectively called the optimistic multi-granulation fuzzy lower approximation and fuzzy upper approximation of X over these dominate relations $R_{A_i}^{\succcurlyeq}(i = 1, \cdots, m)$. X is a multi-granulation fuzzy rough set over these dominate relations $R_{A_i}^{\succcurlyeq}(i = 1, \cdots, m)$ if and only if $\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}(X) \neq \overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}(X)$. Otherwise, X is a multi-granulation fuzzy definable set.

It can be found that the OMGFRS over the dominate relations $R_{A_i}^{\succcurlyeq}(i = 1, \cdots, m)$ will be degenerated into fuzzy rough set when $A_i = A_j$, $i \neq j$ and $[u]_{A_i}^{\succcurlyeq}$ are equivalence classes with respect to the subsets of attributes $A_i(i = 1, \cdots, m)$. That is to say, a fuzzy rough set model is a special instance of the OMGFRS over these dominate relations. If these relations $R_{A_i}^{\succcurlyeq}(i = 1, \cdots, m)$ are equivalence relations and the set A is a classical set, however, the OMGFRS will be degenerated into optimistic multi-granulation rough set [8].

In the following, we employ an example to illustrate the above concepts.

Example 3.1 Suppose Table 3.1 is an ordered information system about the achievements of some students, $U = \{x_1, x_2, \cdots, x_{10}\}$ is a universe which consists of 10 students in some college, *MA (Mathematic), EN (English), MO (Morality), PH (Physical)* are the conditional attributes of the system, and the dominant preference are follows: A≥B≥C≥D. *D (Decision)* is the result of excellent students by the experts according to the achievements of these students.

Table 3.1 An ordered information system about the achievements

Course	MA	EN	MO	PH	D
x_1	85	90	B	A	0.8
x_2	86	90	B	B	0.9
x_3	90	87	A	C	0.7
x_4	88	86	C	D	0.3
x_5	87	85	D	B	0.4
x_6	86	87	A	C	0.6
x_7	84	83	B	A	0.3
x_8	88	88	A	C	0.6
x_9	87	85	C	B	0.2
x_{10}	89	88	B	D	0.7

However, we often face the phenomenon that some universities may give some conditions of excellent students as following:

Condition 1: Not only the marks is higher, but also the morality is better;

Condition 2: Not only the marks is higher , but also the health is better.

We can find that the decision from the table 3.1 is a fuzzy set, and it is easy to find out that $A = (0.8, 0.9, 0.7, 0.3, 0.4, 0.6, 0.3, 0.6, 0.2, 0.7)$ is a fuzzy set of the excellent students which is concluded by the experts.

According to Condition 1 and Condition 2, we can obtain two dominant relations denoted as R_1, R_2.

$$R_1 = \begin{pmatrix} 1&0&0&0&0&0&1&0&0&0\\ 1&1&0&0&0&0&1&0&0&0\\ 0&0&1&1&1&1&1&0&1&0\\ 0&0&0&1&1&0&0&0&1&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&1&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&1&1&1&1&1&1&0\\ 0&0&0&0&1&0&0&0&1&0\\ 0&0&0&1&1&0&1&0&1&1 \end{pmatrix}, R_2 = \begin{pmatrix} 1&0&0&0&0&0&1&0&0&0\\ 0&1&0&0&0&1&0&0&0&0\\ 0&0&1&1&0&1&0&0&1&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&1&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&1&0&1&0&1&0&0\\ 0&0&0&0&1&0&0&0&1&0\\ 0&0&0&1&0&0&0&0&0&1 \end{pmatrix}.$$

According to these conditions, we can raise some questions now.

Question 1: If we consider one of these conditions at least, what is the degree of these students who must be excellent ?

Question 2: When we consider both of these conditions, what is the degree of these students who may be excellent ?

Now, we use the definition of the OMGFRS to solve the above questions.

According to Definition 3.1, we can have

$$\underline{OM^{\succcurlyeq}_{1+2}}(A) = (0.3, 0.6, 0.2, 0.3, 0.4, 0.6, 0.3, 0.3, 0.2, 0.3),$$

$$\overline{OM^{\succcurlyeq}_{1+2}}(A) = (0.8, 0.9, 0.7, 0.3, 0.4, 0.6, 0.3, 0.6, 0.4, 0.4).$$

We can find out that the degree of these students who must be excellent is $0.3, 0.6, 0.2, 0.3, 0.4, 0.6, 0.3, 0.3, 0.2, 0.3$, when we consider at least one of these conditions. And the degree of these students who may be excellent is $0.8, 0.9, 0.7, 0.3, 0.4, 0.6, 0.3, 0.6, 0.4, 0.4$ if we consider both of these conditions.

Just from Definition 3.1, we can obtain some properties of the OMGFRS in an ordered information system.

Proposition 3.1 Let $\mathcal{I}^{\succcurlyeq} = (U, AT, V, f)$ be an ordered information system, $A_1, A_2, \cdots, A_m \subseteq AT$ and $X \in F(U)$. Then the following properties hold.

(1) $\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}(X) \subseteq X$,

(2) $\overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}(X) \supseteq X$;

(3) $\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}(\sim X) =\sim \overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}(X)$;

(4) $\overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}(\sim X) =\sim \underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}(X)$;

(5) $\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}(U) = \overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}(U) = U$;

(6) $\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}(\emptyset) = \overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}(\emptyset) = \emptyset$;

(7) $\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}(X) \supseteq \underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}(\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}(X))$;

(8) $\overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}(X) \subseteq \overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}(\overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}(X))$.

Proof: According to the inductive approach , we only need to prove these properties in the ordered information system which has two dominance relations($A, B \subseteq AT$). It is obvious that all terms hold when $A = B$. When $A \neq B$, the proposition can be proved as follows.

(1) For any $u \in U$ and $A,\ B \subseteq AT$, since $\underline{R_A^{\succcurlyeq}}(X) \subseteq X$ and $\underline{R_B^{\succcurlyeq}}(X) \subseteq X$, we know

$$\wedge\{X(v) \mid v \in [u]_A^{\succcurlyeq}\} \leq X(u)$$

and

$$\wedge\{X(v) \mid v \in [u]_B^{\succcurlyeq}\} \leq X(u)$$

Therefore, $\{\wedge\{X(v) \mid v \in [u]_A^{\succcurlyeq}\}\} \vee \{\wedge\{X(v) \mid v \in [u]_B^{\succcurlyeq}\}\} \leq X(u)$. i.e., $\underline{OM^{\succcurlyeq}_{A+B}}(X) \subseteq X$.

(2) For any $u \in U$ and $A,\ B \subseteq AT$, since $X \subseteq \overline{R_A^{\succcurlyeq}}(X)$ and $X \subseteq \overline{R_B^{\succcurlyeq}}(X)$, we know

$$X(u) \leq \vee\{X(v) \mid v \in [u]_A^{\succcurlyeq}\}$$

and

$$X(u) \leq \vee\{X(v) \mid v \in [u]_B^{\succcurlyeq}\}$$

Therefore, $X(u) \leq \{\vee\{X(v) \mid v \in [u]_A^{\succcurlyeq}\}\} \wedge \{\vee\{X(v) \mid v \in [u]_B^{\succcurlyeq}\}\}$. i.e., $X \subseteq \overline{OM^{\succcurlyeq}_{A+B}}(X)$.

(3) For any $u \in U$ and $A,\ B \subseteq AT$, we have

$$\begin{aligned} &\underline{OM^{\succcurlyeq}_{A+B}}(\sim X)(u) \\ =&\{\wedge\{1 - X(v) \mid v \in [u]_A^{\succcurlyeq}\}\} \vee \{\wedge\{1 - X(v) \mid v \in [u]_B^{\succcurlyeq}\}\} \\ =&\{1 - \vee\{X(v) \mid v \in [u]_A^{\succcurlyeq}\}\} \vee \{1 - \vee\{X(v) \mid v \in [u]_B^{\succcurlyeq}\}\} \\ =&1 - \{\vee\{X(v) \mid v \in [u]_A^{\succcurlyeq}\}\} \wedge \{\vee\{X(v) \mid v \in [u]_B^{\succcurlyeq}\}\} \\ =&\sim \overline{OM^{\succcurlyeq}_{A+B}}(X)(u). \end{aligned}$$

(4) According to $\underline{OM^{\succcurlyeq}_{A+B}}(\sim X) =\sim \overline{OM^{\succcurlyeq}_{A+B}}(X)$, we can have $\underline{OM^{\succcurlyeq}_{A+B}}(X) =\sim \overline{OM^{\succcurlyeq}_{A+B}}(\sim X)$. So it can be found that $\overline{OM^{\succcurlyeq}_{A+B}}(\sim X) =\sim \underline{OM^{\succcurlyeq}_{A+B}}(X)$.

(5) Since for any $u \in U$, $U(u) = 1$, then for any $A, B \subseteq U$,

$$\begin{aligned} &\underline{OM^{\succcurlyeq}_{A+B}}(U)(u) \\ =&\{\wedge\{U(v) \mid v \in [u]_A^{\succcurlyeq}\}\} \vee \{\wedge\{U(v) \mid v \in [u]_B^{\succcurlyeq}\}\} \\ =&1 \\ =&U(u) \end{aligned}$$

and

$$\begin{aligned}&\overline{OM^{\succcurlyeq}_{A+B}}(U)(u)\\=&\{\vee\{U(v)\mid v\in[u]^{\succcurlyeq}_{A}\}\}\wedge\{\vee\{U(v)\mid v\in[u]^{\succcurlyeq}_{B}\}\}\\=&1\\=&U(u).\end{aligned}$$

So $\underline{OM^{\succcurlyeq}_{A+B}}(U)=\overline{OM^{\succcurlyeq}_{A+B}}(U)=U$.

(6) From the duality of the approximation operators in (3) and (4), it is easy to prove $\underline{OM^{\succcurlyeq}_{A+B}}(\emptyset)=\overline{OM^{\succcurlyeq}_{A+B}}(\emptyset)=\emptyset$ by property (5).

(7) Since $\underline{OM^{\succcurlyeq}_{A+B}}(X)\subseteq X$, then for any $u\in U$, one has $\underline{OM^{\succcurlyeq}_{A+B}}(X)(u)\leq X(u)$. So by Definition 3.1, one can obtain $\underline{OM^{\succcurlyeq}_{A+B}}(X)\supseteq\underline{OM^{\succcurlyeq}_{A+B}}(\underline{OM^{\succcurlyeq}_{A+B}}(X))$.

(8) This item can be proved similarly to (7) by the property $\overline{OM^{\succcurlyeq}_{A+B}}(X)\supseteq X$. ■

Proposition 3.2 Let $\mathcal{I}^{\succcurlyeq}=(U,AT,V,f)$ be an ordered information system, $A_1,A_2,\cdots,A_m\subseteq AT$ and $X,Y\in F(U)$. Then the following properties hold.

(1) $\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m}A_i}}(X\cap Y)\subseteq\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m}A_i}}(X)\cap\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m}A_i}}(Y)$,

(2) $\overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m}A_i}}(X\cup Y)\supseteq\overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m}A_i}}(X)\cup\overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m}A_i}}(Y)$;

(3) $X\subseteq Y\Rightarrow\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m}A_i}}(X)\subseteq\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m}A_i}}(Y)$,

(4) $X\subseteq Y\Rightarrow\overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m}A_i}}(X)\subseteq\overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m}A_i}}(Y)$;

(5) $\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m}A_i}}(X\cup Y)\supseteq\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m}A_i}}(X)\cup\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m}A_i}}(Y)$;

(6) $\overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m}A_i}}(X\cap Y)\subseteq\overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m}A_i}}(X)\cap\overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m}A_i}}(Y)$.

Proof: Similar to the proposition 3.1, we only need to prove these properties in the ordered information system which has two dominance relations($A,B\subseteq AT$). All terms hold obviously when $A=B$ or $X=Y$. If $A\neq B$ and $X\neq Y$, the proposition can be proved as follows.

(1) For any $u\in U$, $A,B\subseteq AT$ and $X,Y\in F(U)$,

$$\begin{aligned}&\underline{OM^{\succcurlyeq}_{A+B}}(X\cap Y)(u)\\=&\{\wedge\{(X\cap Y)(v)\mid v\in[u]^{\succcurlyeq}_{A}\}\}\vee\\&\{\wedge\{(X\cap Y)(v)\mid v\in[u]^{\succcurlyeq}_{B}\}\}\\=&\{\wedge\{X(v)\wedge Y(v)\mid v\in[u]^{\succcurlyeq}_{A}\}\}\vee\\&\{\wedge\{X(v)\wedge Y(v)\mid v\in[u]^{\succcurlyeq}_{B}\}\}\\=&\{\underline{R^{\succcurlyeq}_{A}}(X)(u)\wedge\underline{R^{\succcurlyeq}_{A}}(Y)(u)\}\vee\{\underline{R^{\succcurlyeq}_{B}}(X)(u)\wedge\underline{R^{\succcurlyeq}_{B}}(Y)(u)\}\\\leq&\{\underline{R^{\succcurlyeq}_{A}}(X)(u)\vee\underline{R^{\succcurlyeq}_{B}}(X)(u)\}\wedge\{\underline{R^{\succcurlyeq}_{A}}(Y)(u)\vee\underline{R^{\succcurlyeq}_{B}}(Y)(u)\}\\=&\underline{OM^{\succcurlyeq}_{A+B}}(X)(u)\wedge\underline{OM^{\succcurlyeq}_{A+B}}(Y)(u).\end{aligned}$$

Then $\underline{OM^{\succcurlyeq}_{A+B}}(X\cap Y)\subseteq\underline{OM^{\succcurlyeq}_{A+B}}(X)\cap\underline{OM^{\succcurlyeq}_{A+B}}(Y)$.

(2) Similarly, for any $u\in U$, $A,B\subseteq AT$ and $X,Y\in F(U)$,

$$\begin{aligned}&\overline{OM^{\succcurlyeq}_{A+B}}(X\cup Y)(u)\\=&\{\vee\{(X\cup Y)(v)\mid v\in[u]^{\succcurlyeq}_{A}\}\}\wedge\\&\{\vee\{(X\cup Y)(v)\mid v\in[u]^{\succcurlyeq}_{B}\}\}\\=&\{\vee\{X(v)\vee Y(v)\mid v\in[u]^{\succcurlyeq}_{A}\}\}\wedge\\&\{\vee\{X(v)\vee Y(v)\mid v\in[u]^{\succcurlyeq}_{B}\}\}\\=&\{\overline{R^{\succcurlyeq}_{A}}(X)(u)\vee\overline{R^{\succcurlyeq}_{A}}(Y)(u)\}\wedge\{\overline{R^{\succcurlyeq}_{B}}(X)(u)\vee\overline{R^{\succcurlyeq}_{B}}(Y)(u)\}\\\geq&\{\overline{R^{\succcurlyeq}_{A}}(X)(u)\wedge\overline{R^{\succcurlyeq}_{B}}(X)(u)\}\vee\{\overline{R^{\succcurlyeq}_{A}}(Y)(u)\wedge\overline{R^{\succcurlyeq}_{B}}(Y)(u)\}\\=&\overline{OM^{\succcurlyeq}_{A+B}}(X)(u)\vee\overline{OM^{\succcurlyeq}_{A+B}}(Y)(u).\end{aligned}$$

Then $\overline{OM^{\succcurlyeq}_{A+B}}(X\cup Y)\supseteq\overline{OM^{\succcurlyeq}_{A+B}}(X)\cup\overline{OM^{\succcurlyeq}_{A+B}}(Y)$.

(3) Since for any $u\in U$, we have $X(u)\leq Y(u)$. Then the properties hold obviously by Definition 3.1.

(4) The properties can be proved as (3).

(5) Since $X\subseteq X\cup Y$, and $Y\subseteq X\cup Y$, then $\underline{OM^{\succcurlyeq}_{A+B}}(X)\subseteq\underline{OM^{\succcurlyeq}_{A+B}}(X\cup Y)$ and $\underline{OM^{\succcurlyeq}_{A+B}}(Y)\subseteq\underline{OM^{\succcurlyeq}_{A+B}}(X\cup Y)$. So the property $\underline{OM^{\succcurlyeq}_{A+B}}(X\cup Y)\supseteq\underline{OM^{\succcurlyeq}_{A+B}}(X)\cup\underline{OM^{\succcurlyeq}_{A+B}}(Y)$ obviously holds.

(6) This item can be proved similarly as (5) by (4).

The proposition was proved. ■

The fuzzy lower and fuzzy upper approximation in Definition 3.1 are a pair of fuzzy sets. If we associate the cut set of the fuzzy sets, we can make a description of a fuzzy set X by a classical set in an ordered information system.

Definition 3.2 Let $\mathcal{I}^{\succcurlyeq}=(U,AT,V,f)$ be an ordered information system, $A_1,A_2,\cdots,A_m\subseteq AT$ and $X\in F(U)$. For any $0<\beta\leq\alpha\leq1$, the fuzzy lower approximation $\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m}A_i}}(X)$ and fuzzy upper approximation $\overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m}A_i}}(X)$ of X about the α, β cut sets over the dominate relations $R^{\succcurlyeq}_{A_i}(i=1,\cdots,m)$ are defined, respectively, as follows

$$\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m}A_i}}(X)_\alpha=\{u\in U\mid\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m}A_i}}(X)(u)\geq\alpha\},$$

$$\overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m}A_i}}(X)_\beta=\{u\in U\mid\overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m}A_i}}(X)(u)\geq\beta\}.$$

$\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m}A_i}}(X)_\alpha$ can be explained as the set of objects in U which surely belong to X over the dominance relations $R^{\succcurlyeq}_{A_i}(i=1,\cdots,m)$ and the memberships of which are more than α, while $\overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m}A_i}}(X)_\beta$ is the set of objects in U which possibly belong to X over the dominance relations $R^{\succcurlyeq}_{A_i}(i=1,\cdots,m)$ and the memberships of which are more than β.

Proposition 3.3 Let $\mathcal{I}^{\succcurlyeq}=(U,AT,V,f)$ be an ordered information system, $A_1,A_2,\cdots,A_m\subseteq AT$ and $X,Y\subseteq F(U)$. For any $0<\beta\leq\alpha\leq1$, we have

(1) $\underline{\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}}(X\cap Y)_\alpha \subseteq \underline{\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}}(X)_\alpha \cap \underline{\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}}(Y)_\alpha$,

(2) $\overline{\overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}}(X\cup Y)_\beta \supseteq \overline{\overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}}(X)_\beta \cup \overline{\overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}}(Y)_\beta$;

(3) $X \subseteq Y \Rightarrow \underline{\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}}(X)_\alpha \subseteq \underline{\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}}(Y)_\alpha$,

(4) $X \subseteq Y \Rightarrow \overline{\overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}}(X)_\beta \subseteq \overline{\overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}}(Y)_\beta$;

(5) $\underline{\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}}(X\cup Y)_\alpha \supseteq \underline{\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}}(X)_\alpha \cup \underline{\underline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}}(Y)_\alpha$,

(6) $\overline{\overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}}(X\cap Y)_\beta \subseteq \overline{\overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}}(X)_\beta \cap \overline{\overline{OM^{\succcurlyeq}_{\sum_{i=1}^{m} A_i}}}(Y)_\beta$.

Proof: It is easy to prove by Definition 3.2 and Proposition 3.2. ■

IV. Conclusion

The theories of rough set and fuzzy set both extended the classical set theory in terms of dealing with uncertainty and imprecision. However, the theory of fuzzy set pay more attention to the fuzziness of knowledge while the theory of rough set to the roughness of knowledge. For the complement of the two types of theory, fuzzy rough set models are investigated to solve practical problem. Given that the equivalence relation in the fuzzy rough set theory is too rigorous for some practical application, it is necessary to weaken the equivalence relation to dominance relation. The contribution of this paper is having constructed a kind of fuzzy rough set over dominance relations associated with granular computing called optimistic multi-granulation fuzzy rough set model over dominance relations, in which the set approximation operators are defined on the basis of multiple dominance relations. What's more, we make conclusion that fuzzy rough set model is a special case of the optimistic multi-granulation fuzzy rough set over dominance relations by analyzing the definition . More properties of the optimistic multi-granulation fuzzy rough set over dominance relations are discussed. In this paper, we only discussed the optimistic multi-granulation fuzzy rough set. We are investigating the pessimistic multi-granulation fuzzy rough set based on dominance relations in our study.

Acknowledgment

The authors would like to thank the anonymous referees for their valuable comments and suggestions. This work is supported by National Natural Science Foundation of China (No. 61105041, 71071124, and 11001227), Post-doctoral Science Foundation of China (No. 20100481331), Natural Science Foundation Project of CQ CSTC (No. cstc2011jjA40037), Graduate Innovation Foundation of Chongqing University of Technology (No.YCX2011312).

References

[1] D. Dubois and H. Prade, *Putting rough sets and fuzzy sets together*, 1992.

[2] S. Greco, B. Matarazzo and R. Slowinski, *Dominance-based Rough Set approach as a proper way of handling graduality in rough set theory*, 2007.

[3] J. R. Hobbs, "Granularity", *In Proceedings of the Ninth International Joint Conference on Artificial Intelligence*, pp. 432-435, 1985.

[4] M.A. Khan and M.H. Ma, "A Model Logic for Multiple-Source Tolerance Approximation Space", *Logic and its Applications*, 6521, pp. 124-136, 2011.

[5] L.A. Zadeh, *Fuzzy Sets and Information Granularity*, 1979.

[6] L.A. Zadeh, "Some reflections on soft computing, granular computing and their roles in the conception, design and utilization of information/intelligent systems", *Soft Computing - A Fusion of Foundations, Methodologies and Applications*, 2, pp.23-25, 1998.

[7] Z. Pawlak, "Rough sets", *International Journal of Computer and Information Sciences*, 11 , pp. 341-356, 1982.

[8] Y. H. Qian, J. Y. Liang, Y.Y. Yao and C.H. Dang, "MGRS: A multi-granulation rough set", *Information Sciences*, 180 , pp. 949-970, 2010.

[9] W.H. Xu, X.Y. Zhang and W.X. Zhang, "Knowledge granulation, knowledge entropy and knowledge uncertainty measure in ordered information systems", *Applied soft computing*, 9 , pp. 1244-1252, 2009.

[10] W.H. Xu, X.Y. Zhang and J.M. Zhong, "Attribute reduction in ordered information system based on evidence theory", *Knowledge and Information Systems*, 25 , pp. 198-184, 2010 .

[11] W.H. Xu, Q.R. Wang and X.T. Zhang, Multi-granulation fuzzy rough set model on tolerance relations, *Fourth International Workshop on Advanced Computational Intelligence*, pp.359-366, 2011 .

[12] W.H. Xu, Q.R. Wang and X.T. Zhang, Multi-granulation Fuzzy Rough Sets in a Fuzzy Tolerance Approximation Space, *International Journal of Fuzzy Systems*, 13(4), pp. 246-259, 2011.

[13] W.H. Xu, W.X. Sun , X.Y. Zhang and W.X. Zhang, Multiple granulation rough set approach to ordered information systems, *International Journal of General Systems*, 41(5), pp. 475-501, 2012.

[14] W.X. Zhang, J.Y. Liang and W.Z. Wu, *Fuzzy information systems and knowledge discovery in: Information systems and knowledge discovery*, 2003.

[15] W.X. Zhang, W.Z.Wu, J.Y. Liang and D.Y.Li, *Theory and method of rough sets*, 2001.